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Summary

Learning Objectives:

From this module students may get to know about the following:

1. The alternative method of writing the energy in the electrostatic case in terms of the field rather than the charges is explained.
2. Exactly in the same fashion it is explained as to how the energy in a current loop can be thought of as being stored in the magnetic field.
3. The general case of electrodynamics involving changing electromagnetic fields is taken up and the law of conservation of energy is derived.

5. Conservation laws – I

In this and the next module we look at the various classical conservation laws for a system of charges and electromagnetic fields. We will see that for a proper understanding of the laws of conservation of energy, momentum and angular momentum etc., we must endow the electromagnetic field itself with these attributes. Thus the field becomes as real an entity as a particle, carrying its own energy and momentum etcetera. In this module we will look at the law of conservation of energy, known in this context as the Poynting theorem, and study momentum and angular momentum conservation in the next module. We begin by looking at the some of the well known results from electrostatics and steady currents and recast them in a form which is more useful for our present studies.

5.1 Energy in electrostatic field

Let there be a system of n point charges. If the charge on the i^th^ particle is q_i, then the total electrostatic energy of the system is given by

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{1}$$

For a continuous charge distribution the potential energy takes the form

$$W = \frac{1}{8\pi\varepsilon_0} \iint \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x\, d^3x' \tag{2}$$

Here $\rho(\vec{x})$ is the charge density at the point $\vec{x}$, and the integration is, in principle, over all space. However $\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x})}{|\vec{x} - \vec{x}'|} d^3x'$ is the scalar potential $\Phi(\vec{x})$. Hence the above expression for the potential energy can be written in the alternative form

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3x \, .$$

All these expressions for the electrostatic potential energy are expressed in terms of the position of the charges and therefore emphasize the fact that the energy has arisen from the Coulomb force. An alternative approach is to express the energy in terms of the electric field of the system.

Use Coulomb law $\vec{\nabla}.\vec{E} = \rho / \varepsilon_0$ to eliminate ρ from the above expression and obtain

$$W = \frac{\varepsilon_0}{2} \int (\vec{\nabla}.\vec{E}) \Phi(\vec{x}) d^3x = \frac{\varepsilon_0}{2} \int [\vec{\nabla}.(\Phi \vec{E}) - \vec{E}.\vec{\nabla}\Phi] d^3x$$

The first integral can be converted into a surface integral by use of Gauss theorem. If, as is always the case, the fields fall off sufficiently rapidly, the surface integral vanishes. In the second integral use $\vec{E} = -\vec{\nabla}\Phi$ and we get

$$W = \frac{\varepsilon_0}{2}\int E^2 d^3x = \int w_{el}(\vec{x})d^3x \quad . \tag{3}$$

$$w_{el}(\vec{x}) = \frac{\varepsilon_0}{2}E^2 \ . \tag{4}$$

The quantity $w_{el}(\vec{x})$ can be regarded as the energy density at the point $\vec{x}$. In this expression the energy is expressed in terms of the electric field rather than the charge or charge density. True the field is produced by the charges but there is no direct reference to them. This provides an alternative interpretation, viz., the energy is stored in the electrostatic field rather than in the charges. The regions that contribute to the integral may be altogether different in the two cases. For example, for a spherical shell, the charge is confined to the shell, whereas the field is present everywhere outside this surface. Nevertheless, in electrostatics it really does not matter much as to which interpretation we choose to adopt. The situation, as we will see later, is different for the general case of time-varying fields.

Another point that needs clarification at this stage is the following: The energy obtained from the expression in terms of an integral is different from the one in terms of a discrete sum. In particular, whereas the former is always positive, the latter can be positive or negative depending on the signs of the charges. The reason for this difference is the self-energy of the charges. Whereas the self energy contribution is included in the expression in terms of the integral, it is not included in the double sum expression, since the terms i = j are explicitly excluded. The self energy is a tricky business and needs very careful handling. In fact, for a proper understanding of the self-energy problem one has to take recourse to quantum formulation of electrodynamics; even in quantum electrodynamics many questions of a fundamental nature remain unresolved. From a practical point of view, though expression (3) is more complete, expression (1) is more appropriate for point charge distributions.

5.2 Energy in magnetic field

Just as it takes certain amount of energy to create a charge distribution, it takes energy to establish a current in a circuit. This is the work done against the back emf. This is the "potential energy" stored in the current loop. Suppose for a moment that we have a single circuit with a constant current I flowing through it. A change of flux in the circuit induces an electromotive force ε around it. The work done on a unit charge against the emf is $-\varepsilon$. The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt}. \tag{5}$$

Here L is the self inductance of the coil. The total work done in building up the current from zero to I is, on integrating,

$$W = \frac{1}{2}LI^2. \tag{6}$$

Precisely as in the case of electrostatics, this expression can be put in an alternative form which is more illuminating. The flux through a loop is given by

$$\Phi = \int_S \vec{B}.d\vec{a} = \int_S (\vec{\nabla} \times \vec{A}).d\vec{a} = \oint_C \vec{A}.d\vec{l}\ . \tag{7}$$

In the last equation the line integral is over the perimeter of the loop and is obtained by the use of Stokes's theorem. The flux for a loop through which a current I is passing is LI. Hence

$$LI = \oint_C \vec{A}.d\vec{l}\ , \tag{8}$$

and therefore

$$W = \frac{1}{2} I \oint_C \vec{A}.d\vec{l} \ . \qquad (9)$$

Since the current I is along the loop, $Id\vec{l} = \vec{I}dl$, and the above equation can be written as

$$W = \frac{1}{2} \oint_C \vec{A}.\vec{I}dl \qquad (11)$$

Now the generalization to volume currents is clear:

$$W = \frac{1}{2} \int_V \vec{A}.\vec{J}d^3x \qquad (12)$$

Following the same line of reasoning as in electrostatics, we express W entirely in terms of the magnetic field. Using Ampere's law (remember we are doing steady currents) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, we can eliminate $\vec{J}$ to obtain

$$W = \frac{1}{2\mu_0} \int_V \vec{A}.(\vec{\nabla} \times \vec{B})d^3x \qquad (13)$$

On using the vector identity

$$\vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B}) \qquad (14)$$

the above expression becomes

$$W = \frac{1}{2\mu_0} \int_V [\vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{\nabla}.(\vec{A} \times \vec{B})]d^3x \qquad (15)$$

The second integral can be converted into a surface integral by use of Gauss theorem. For any finite current distribution the fields fall off sufficiently rapidly and the surface integral vanishes. In the first integral use $\vec{B} = \vec{\nabla} \times \vec{A}$ and we get

$$W = \frac{1}{2\mu_0} \int B^2 d^3x = \int w_{mag}(\vec{x}) d^3x \quad . \tag{16}$$

$$w_{mag}(\vec{x}) = \frac{1}{2\mu_0} B^2 \tag{17}$$

This is the magnetic analogue of the expression for electrostatic energy. In this case also we can say, as an alternative view, that the energy is stored in the magnetic field. $w_{mag}(\vec{x})$ is then the energy density of the magnetic field.

5.3 Conservation of energy in electromagnetic field

We now study the law of conservation of energy as applied to a system of charges, currents and electromagnetic fields. For a single charge q, moving with velocity $\vec{v}$ the rate of doing work by external electric and magnetic fields $(\vec{E}, \vec{B})$ is

$$\frac{dW}{dt} = \vec{F} . \frac{d\vec{s}}{dt} = \vec{F} . \vec{v} = q(\vec{E} + \vec{v} \times \vec{B}) . \vec{v} = q\vec{v} . \vec{E} \, . \tag{18}$$

For a system of point charges the corresponding expression will be $\sum q_i \vec{v}_i . \vec{E}_i$ in obvious notation. For continuous charge and current distributions we replace $q\vec{v}$ by the current density $\vec{J}(\vec{x})$ and summation over the charges by integration over the current distribution to obtain

$$\frac{dW}{dt} = \int_V \vec{J}.\vec{E} d^3 x \,. \qquad (19)$$

This power represents a conversion of electromagnetic energy into heat or mechanical work done on the particles. If total energy of the system is to remain constant, it must be compensated by an equal rate of decrease of electromagnetic energy within the volume V. In order to exhibit this conservation law explicitly, the right hand side must be written in terms of the fields alone rather than the current density of the charges. This we do by using Maxwell's equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (20)$$

to eliminate $\vec{J}$ from the right side:

$$\vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (21)$$

On substituting this expression for $\vec{J}$ in (1)

$$\int_V \vec{J}.\vec{E} d^3 x = \int_V [\frac{1}{\mu_0} \vec{E}.(\vec{\nabla} \times \vec{B}) - \varepsilon_0 \vec{E}.\frac{\partial \vec{E}}{\partial t}] d^3 x \qquad (22)$$

Employing now the vector identity

$$\vec{\nabla}.(\vec{E} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{E}) - \vec{E}.(\vec{\nabla} \times \vec{B}) \qquad (23)$$

and the Faraday's law

$$\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

we have

$$\int_V \vec{J}.\vec{E}d^3x=-\int_V[\frac{1}{\mu_0}\vec{\nabla}.(\vec{E}\times\vec{B})+\frac{1}{\mu_0}\vec{B}.(\vec{\nabla}\times\vec{E})+\varepsilon_0\vec{E}.\frac{\partial\vec{E}}{\partial t}]d^3x \qquad (24)$$

or

$$\int_V \vec{J}.\vec{E}d^3x=-\int_V[\frac{1}{\mu_0}\vec{\nabla}.(\vec{E}\times\vec{B})+\frac{1}{\mu_0}\vec{B}.\frac{\partial\vec{B}}{\partial t}+\varepsilon_0\vec{E}.\frac{\partial\vec{E}}{\partial t}]d^3x \qquad (25)$$

We have already demonstrated that $\frac{\varepsilon_0}{2}E^2$ represents the energy density in the electrostatic field and $\frac{1}{2\mu_0}B^2$ the energy density in the magnetic field. We now make the reasonable assumption that the sum of the two quantities $\frac{1}{2\varepsilon_0}E^2+\frac{\mu_0}{2}B^2$ represents the energy density in the electromagnetic field in the case of time-varying fields as well. If we denote the total energy density by

$$u=\frac{1}{2}(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2) \qquad (26)$$

equation (2) can be written as

$$-\int_V \vec{J}.\vec{E}d^3x=\int_V[\frac{\partial u}{\partial t}+\frac{1}{\mu_0}\vec{\nabla}.(\vec{E}\times\vec{B})]d^3x$$

$$= \int_v \frac{\partial u}{\partial t} d^3x + \frac{1}{\mu_0} \oint_s \vec{E} \times \vec{B} . d\vec{a} \tag{27}$$

Here again we have used Gauss theorem to convert the volume integral to a surface integral over the bounding surface. This is the Poynting's theorem.

The first integral on the right is the total energy stored in the electromagnetic field. The second term can be interpreted as the rate at which the total energy is carried out of the volume across its bounding surface. Poynting theorem then says that the negative of the rate of work done by the charges equals the rate at which the energy increases inside the volume plus the rate at which the energy flows out of the volume. The energy per unit area per unit time transported by the fields is called the Poynting vector, and is given by

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \vec{B} / \mu_0 \tag{28}$$

$\vec{S}.d\vec{a}$ is the energy crossing an infinitesimal area $d\vec{a}$ per uni time or energy flux. So $\vec{S}$ is also called the energy flux density.

Since the volume V is arbitrary, this can be cast into the form of a differential equation:

$$\frac{\partial u}{\partial t} + \vec{\nabla} . \vec{S} = -\vec{J} . \vec{E} \ . \tag{29}$$

This is the differential form of the law of conservation of energy, and as usual has the form of a continuity equation.

The work W done on the charge system will increase its energy and appear in the form of heat or mechanical energy. Since matter is ultimately composed of charged particles, we can think of this conversion as the increase of energy of charged particles per unit volume. Then we can interpret Poynting theorem for the microscopic fields as the statement of conservation of energy of the combined system of charges and fields. If we denote the total energy of the particles within the volume V by W_{mech}, and assume that no particles move out of the volume, we have

$$\frac{dW_{mech}}{dt} = \int_V \vec{J}.\vec{E} d^3x \tag{30}$$

Then Poynting theorem expresses the conservation of energy of the combined system:

$$\frac{dW}{dt} = \frac{d}{dt}(W_{mech} + W_{field}) = -\oint_S \hat{n}.\vec{S} da = -\int_V \vec{\nabla}.\vec{S} d^3x \tag{31}$$

The total field energy within the volume V is

$$W_{field} = \int_V u_{field} d^3x = \frac{1}{2}\int_V (\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2) d^3x \tag{32}$$

Writing in terms of energy densities, if u_{mech} is the mechanical energy density, the above equation takes the form

$$\frac{d}{dt}\int_V u d^3x = \frac{d}{dt}\int_V (u_{mech} + u_{field}) d^3x = -\int_V \vec{\nabla}.\vec{S} d^3x \tag{33}$$

Since volume V is arbitrary, this gives the differential form of Poynting theorem:

$$\frac{\partial u}{\partial t} = \frac{\partial (u_{mech} + u_{field})}{\partial t} = -\vec{\nabla}.\vec{S} \ . \tag{34}$$

Summary

1. In this module we have introduced the student to an alternative method of writing the energy in the electrostatic case in terms of the field rather than the charges.
2. Next, exactly in the same fashion it is explained as to how the energy in a current loop can be thought of as being stored in the magnetic field.
3. The general case of electrodynamics involving changing electromagnetic fields is taken up and the law of conservation of energy, called Poynting theorem, is derived.
4. It is explained that for a proper understanding of the law of conservation at a local level it is essential that the energy be assigned to the electromagnetic field.